

# 1 Boundary Value Problems

The notion of *boundary value problems* applies to a wide variety of topics in physics, differential equations, and engineering. Here we address the concept of fitting a curve  $f(x)$  not simply to discrete data points, but to differentiable continua (points on curves). Then, constraints on derivatives of  $f(x)$  have a general effect on the way  $f(x)$  is optimized.

## 1.1 A Two-Roads Problem

One example of a simple boundary-value problem begins with two parabolic non-intersecting roads or train tracks given by

$$y_1(x) = -\frac{x^2}{4} - 1 \qquad y_2(x) = \frac{x^2}{4} + 1.$$

Next, suppose we are required to find a function  $f(x)$  that, in the smoothest way possible, connects the point  $(-2, -2)$  on the  $y_1$ -curve to the point  $(1, 5/4)$  on the  $y_2$ -curve.

Let us proceed by guessing that the solution, whatever it is, can be approximated by an  $N$ -order polynomial

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots = \sum_{j=1}^N A_j x^{j-1},$$

where some choice of coefficients  $\{A_j\}$  contains the correct answer. The most obvious facts we can write about  $f(x)$  is it must match  $y_1(-2)$  and  $y_2(1)$  as given in the problem statement. This means we can generate two equations:

$$\begin{aligned} y_1(-2) = f(-2) = -2 &= A - 2B + 4C - 8D + 16E - \dots \\ y_2(1) = f(1) = 5/4 &= A + B + C + D + E + \dots \end{aligned}$$

So far so good, but there are still more unknowns than equations - the system is under-determined without new information.

Looking again at the functions  $y_{1,2}(x)$ , we can differentiate each twice before reaching zero:

$$\begin{aligned} \frac{d}{dx}y_1(x) &= -\frac{x}{2} & \frac{d}{dx}y_2(x) &= \frac{x}{2} \\ \frac{d^2}{dx^2}y_1(x) &= -\frac{1}{2} & \frac{d^2}{dx^2}y_2(x) &= \frac{1}{2} \end{aligned}$$

Evaluating each of these at the boundary points  $(-2, -2)$  and  $(1, 5/4)$ , we can calculate a number for the slope and the second derivative at each boundary point

$$\begin{aligned} \frac{d}{dx}y_1(-2) &= 1 & \frac{d}{dx}y_2(1) &= \frac{1}{2} \\ \frac{d^2}{dx^2}y_1(-2) &= -\frac{1}{2} & \frac{d^2}{dx^2}y_2(1) &= \frac{1}{2}, \end{aligned}$$

where higher-order derivatives are clearly zero. Meanwhile, we may also take the first and second derivatives of our guess function  $f(x)$ , giving:

$$\begin{aligned}\frac{d}{dx}f(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \dots \\ &= \sum_{j=2}^N (j-1) A_j x^{j-2} \\ \frac{d^2}{dx^2}f(x) &= 2C + 3 \cdot 2Dx + 4 \cdot 3Ex^2 + 5 \cdot 4Fx^3 + 6 \cdot 5Gx^4 \dots \\ &= \sum_{j=3}^N (j-1)(j-2) A_j x^{j-3}\end{aligned}$$

We already know that the function  $f(x)$  matches the curves  $y_{1,2}$  at the boundary points. Now comes the crucial observation: the function *and its derivatives* match the curves *and their derivatives* at the boundary points. This means we gain four new equations

$$\begin{aligned}\frac{d}{dx}(y_1(-2)) &= \frac{d}{dx}(f(-2)) = 1 \\ \frac{d}{dx}(y_2(1)) &= \frac{d}{dx}(f(1)) = \frac{1}{2} \\ \frac{d^2}{dx^2}(y_1(-2)) &= \frac{d^2}{dx^2}(f(-2)) = -\frac{1}{2} \\ \frac{d^2}{dx^2}(y_2(1)) &= \frac{d^2}{dx^2}(f(1)) = \frac{1}{2},\end{aligned}$$

where in terms of the unknown coefficients, the above read:

$$\begin{aligned}1 &= B + 2C(-2) + 3D(-2)^2 + 4E(-2)^3 + 4F(-2)^4 + \dots \\ 1/2 &= B + 2C + 3D + 4E + 4F + \dots \\ -1/2 &= 2C + 3 \cdot 2D(-2) + 4 \cdot 3E(-2)^2 + 5 \cdot 4F(-2)^3 + 6 \cdot 5G(-2)^4 + \dots \\ 1/2 &= 2C + 3 \cdot 2D + 4 \cdot 3E + 5 \cdot 4F + 6 \cdot 5G + \dots\end{aligned}$$

Interestingly, we now have six total equations, up from two. This should mean if we choose  $f(x)$  to contain six unknowns, which amounts to keeping the  $A$ - through  $F$ -terms, and discarding any  $G$ - or higher terms, then the system left is indeed determined, and can be solved exactly. Truncating the series accordingly, all information in the problem is contained in the augmented matrix

$$\begin{bmatrix} 1 & -2 & 4 & -8 & 16 & -32 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 5/4 \\ 0 & 1 & -4 & 12 & -32 & 80 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 1/2 \\ 0 & 0 & 2 & -12 & 48 & -160 & -1/2 \\ 0 & 0 & 2 & 6 & 12 & 20 & 1/2 \end{bmatrix},$$

where Gaussian elimination hands us the solution as right-most column in the RREF matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 43/81 \\ 0 & 1 & 0 & 0 & 0 & 0 & 94/81 \\ 0 & 0 & 1 & 0 & 0 & 0 & -149/324 \\ 0 & 0 & 0 & 1 & 0 & 0 & -23/162 \\ 0 & 0 & 0 & 0 & 1 & 0 & 19/162 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7/162 \end{bmatrix}.$$

Finally, the curve is given by

$$f(x) = \frac{43}{81} + \frac{94}{81}x - \frac{149}{324}x^2 - \frac{23}{162}x^3 + \frac{19}{162}x^4 + \frac{7}{162}x^5 \quad -2 \leq x \leq 1$$

as shown.

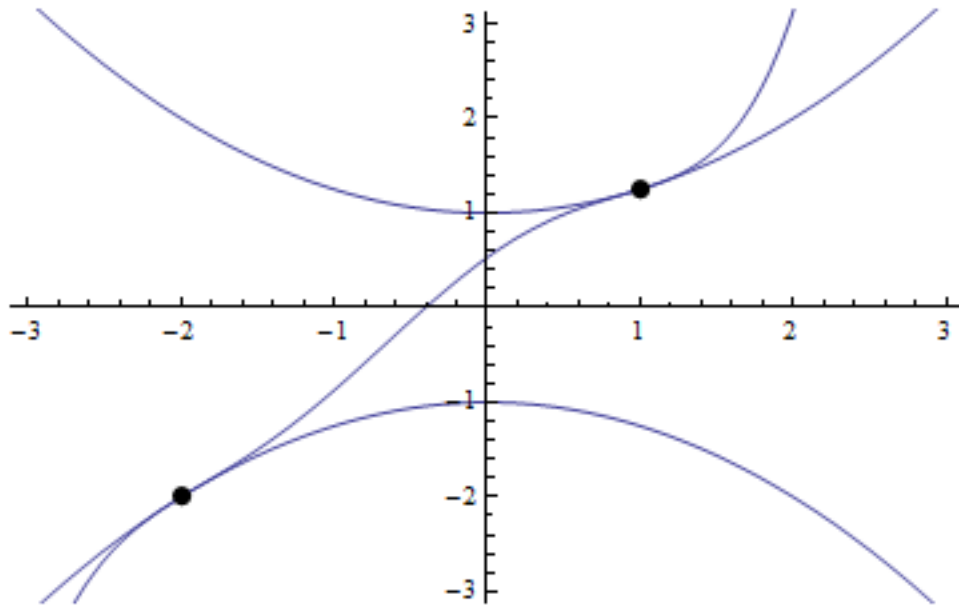


Figure 1: Parabolic roads with connecting curve